SUGGESTED SOLUTION TO HOMEWORK 7

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Problem 1. (a) Prove that for every two subspaces X_1 and X_2 of a Hilbert space,

$$(X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp.$$

(b) Prove that for every two closed subspaces X_1 and X_2 of a Hilbert space,

$$(X_1 \cap X_2)^\perp = \overline{X_1^\perp + X_2^\perp}.$$

Proof. (a) On the one hand, for arbitrary $x \in (X_1 + X_2)^\perp$, since $X_1 \subset X_1 + X_2$, then for arbitrary $y \in X_1$, we have $x \perp y$ which implies $x \in X_1^\perp$. Therefore $(X_1 + X_2)^\perp \subset X_1^\perp$. Similarly, $(X_1 + X_2)^\perp \subset X_2^\perp$. Hence $(X_1 + X_2)^\perp \subset X_1^\perp \cap X_2^\perp$.

On the other hand, for arbitrary $y \in X_1 + X_2$, there exist $y_1 \in X_1$ and $y_2 \in X_2$ such that $y = y_1 + y_2$, then for arbitrary $x \in X_1^\perp \cap X_2^\perp$, we have $x \perp y_1$ and $x \perp y_2$, which implies $x \perp y$. Therefore we also have $X_1^\perp \cap X_2^\perp \subset (X_1 + X_2)^\perp$.

Combining the above results, we have $(X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp$.

(b) Since X_1, X_2 are closed, we have $(X_i^\perp)^\perp = X_i$ for $i = 1, 2$. Then from (a), we have $(X_1^\perp + X_2^\perp)^\perp = (X_1^\perp)^\perp \cap (X_2^\perp)^\perp = X_1 \cap X_2$. Therefore $\overline{X_1^\perp + X_2^\perp} = ((X_1^\perp + X_2^\perp)^\perp)^\perp = (X_1 \cap X_2)^\perp$. □

Problem 2. Let P be the vector space of all real polynomials on $[-1, 1]$. Show that

$$\langle x, y \rangle = \int_{-1}^{1} x(t)y(t)dt$$

defines an inner product on P. Use the Gram-Schmidt process to orthonormalize the set $\{1, t, t^2\}$.

Proof. It is clear that $\langle \cdot, \cdot \rangle$ defines an inner product on P. By the Gram-Schmidt process, we find a set of orthonormal vectors,

$$e_1(t) := \frac{\sqrt{2}}{2}, \quad e_2(t) := \frac{\sqrt{6}}{2}t, \quad e_3(t) := \frac{3\sqrt{10}}{4}\left(t^2 - \frac{1}{3}\right).$$

□

Problem 3. Let $T : \ell_2 \to \ell_2$ be defined by

$$T : (x_1, \cdots, x_n, \cdots) \mapsto (x_1, \cdots, \frac{1}{n}x_n, \cdots).$$

Show that $\mathcal{R}(T)$ is not closed in ℓ_2.

Proof. Suppose on the contrary that $\mathcal{R}(T)$ is closed in ℓ_2, since ℓ_2 is a Banach space, we have $\mathcal{R}(T)$ is also a Banach space. Moreover, it is clear that T is bounded and bijective, therefore by the open mapping theorem, we also have T^{-1} is bounded. However, consider $\{e_n\}_{n\geq 1}$ defined by

$$e_n(i) = \begin{cases} 1, & i = n, \\ 0, & i \neq n. \end{cases}$$

Then $\|e_n\|_2 = 1$ and $e_n \in \mathcal{R}(T)$. Since

$$\|T^{-1}\| \geq \|T(e_n)\| = n,$$

therefore by letting n goes to infinity, we have T^{-1} is not bounded which is a contradiction, hence $\mathcal{R}(T)$ is not closed. □

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